The Sets of Real and Complex Numbers Are Denumerable

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***Abstract*— This short article demonstrates that both the sets of real and complex numbers are actually denumerable, which means that those numeric sets, as well as the set of natural numbers, actually have the same cardinality, hence there exists a unique transfinite number.**

***Index Terms*— Cantor, Computability, Enumerability, Foundations of Computer Science, Foundation of Mathematics, Halting Problem, Mathematics, Numbers and Sets, Number Theory, Philosophy, Real, Transfinite.**

Cantor's diagonal process has been recognized as one of the most significant methods, which has been applied to the solution of important problems in computing such as the unsolvability of the halting problem[2]. Although Georg Cantor died in 1918, let us suppose that Cantor is playing a game with a machine M, which is asked for producing a sequence of all real numbers. Whenever M prints such a number, Georg Cantor, by using his own diagonal process, writes another number which is different from the first number of the sequence by the first digit, from the second number of the sequence by the second digit, and so forth. As we know, this set in infinite, but whenever M prints a number we obtain a finite and countable set. In this way, Georg Cantor always beautifully manages to write a new number outside the sequence, no matter what the numbers are which M produces. Thus, he concluded that, by using that finite method, the set of real numbers has greater cardinality than the set of natural numbers because of the transcendental numbers. Nonetheless, using the decimal representation for the reals prevents M to finish writing the first irrational number. In other words, while producing an infinite set by writing a sequence of its elements, there will never be any time for refuting that. A question is whether or not we should permit such a refutation over infinite sets, although his proof is widely accepted. It is also known that $|\mathbb{Q}| = |\mathbb{N}|$ (and also, the set of odd numbers has the same cardinality of the natural numbers), and the reason is simple as explained, for instance, by [3]:

$$\frac{1}{1}, \quad \frac{1}{2}, \frac{2}{1}, \quad \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \quad \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \ \ldots$$

or, diagrammatically as follows:

1st	2nd	4th	7th	11th	16th
3rd	5th	8th	12th	17th	
6th	9th	13th	18th		
10th	14th	...			
15th					

In other words, the above triangular diagram[4] shows the order of the fractions where each entry corresponds to a fraction whose numerator is the row number (abscissa) and the denominator is the column number (ordinate), in order to present all fractions diagonally and then prove $|\mathbb{Q}| = |\mathbb{N}|$. Let us call this ordering technique *triangular* for later reference.

Given a pair of indexes $i, j : \mathbb{N}$, positive ones, perhaps the easiest way of finding the corresponding position in the above one-one correspondence with $\mathbb{N}$ is the following:

$$g(i,j) = \frac{i(i+1)}{2} + 1 - j$$

where g here informs the referred to position in the ordered bijection between the representation of a set and $\mathbb{N}$. Thus, if we have that position n and wish to find the pair of indexes, we just use the following formula:

$$h(n) = \langle\ i,\ j\ \rangle, \text{where...}$$

$$\begin{cases} i = f(0,n) - (g(f(0,n),1) - n) \\ j = g(f(0,n),1) - n + 1 \end{cases}$$

which in its turn uses the f, defined as follows:

$$f(x,y) = \begin{cases} 1 + f(x+1,y) & \text{if } g(x+1,1) \leq y \\ 0 & \text{if } g(x+1,1) > y \end{cases}$$

The same setting holds for $\mathbb{N}^-$, in which numbers can be paired with the corresponding positive ones. In this way, by regarding the sums between the numerators and the corresponding denominators, M can make a one-one correspondence between the rational numbers and the natural numbers. However, while M is printing the numbers of this sequence, that is, while M has printed the (n)th number of the sequence, that is x_n, its opponent can always anticipate and write the following number of the sequence, i.e. x_{n+1}. In accordance with the criticism above, for the sequence is infinite, this means that the opponent of M has a finite method for always showing a new number, different from any previous number of the sequence. However, the rules of this game is at least controversial. What such a procedure refutes is the *arrangement* of the numbers in that sequence, and based on the hypothesis that there exists a kind of right by the person who uses that method to have the final words in that logics game, while the sequence is infinite, i.e. if the game is not of two great moves only but instead of infinite small moves, after the person produces a different number, M can include this number in its sequence, and so forth. In this way, one proves and the other one disproves: that is, the last player wins the (infinite) logics game.

Here, the present proof has the assumption that if a number exists, there exists one or more numeric functions that lead to that instanced number, and that those functions have domains which are the same or more primitive than their images that contain the number. Together with the final words in the logical game, we show a different arrangement and different representation of the real numbers that include the transcendental numbers, and finally the arrangement can be put into (or imagined to be in) a one-one correspondence with $\mathbb{N}$. Firstly, the present author defines the types needed for defining and then building the sequence:

$$\mathbb{L}: \ \mathbb{R} \times \mathbb{L}$$

or, alternatively,

$$\mathbb{L}: \ \epsilon$$

where $\mathbb{R}$ means the set of real numbers whose elements are in a proper representation. Further, $\mathbb{L}$ can be empty (by ϵ) but sequences of type $\mathbb{L}$ will always be finite. A sequence of type $\mathbb{L}$ is informally referred to here as *list*, and the list is finite. We shall need another type definition as the following:

$$\mathcal{M}: \ \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{Q} \times \mathbb{L}$$

Now let us make $S : \mathcal{M}$ be our solution, indexed by two positive natural numbers, i and j, where each application $S(i,j)$ refers to a rational number q. More precisely,

$$q \ = \begin{cases} \frac{i-1}{j-1} & \text{if } j-1 \neq 0 \\ 0 & \text{if } j-1 = 0 \end{cases}$$

and a finite sequence (list) of real numbers. Call this sequence $\mathcal{F}$, i.e. $S(i,j) = \langle q, \mathcal{F} \rangle$ and, for each pair i and j, simply $\mathcal{F}(i,j)$. The formula $S(i,j)_0$ is used to indicate q (and its symmetric value). Furthermore, inside $S(i,j)$, the first element of $\mathcal{F}(i,j)$ is denoted as $elem\langle \mathcal{F}(i,j), 1\rangle$; the second element of $\mathcal{F}(i,j)$ is denoted as $elem\langle \mathcal{F}(i,j), 2\rangle$; and so on, until the last element of $\mathcal{F}(i,j)$, which is $elem\langle \mathcal{F}(i,j), |\mathcal{F}(i,j)|\rangle$. In this way, a real number is represented by a tuple of type $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and of the form $\langle i,j,k \rangle$, where i and j indicate $S(i,j)$, and together with k this 3-tuple indicates the real number in the following way:

$$r = \begin{cases} S(i,j)_0 & \text{if } k = 0 \\ elem(\mathcal{F}(i,j), k) & \text{if } k > 0 \end{cases}$$

Let us refer to each element of S as *cell*, i.e. for every i and every j, $S(i,j)$ is a cell. Furthermore, let (i,j) mean the indexes of the current cell from now on.

In this way, it is known that there exists a finite sequence of arithmetic functions (including all basic operations), typically with arity 1 (that is, $\mathbb{R} \longrightarrow \mathbb{R}$ functions) and 2 (that is, $\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ functions). In any case, a 3-ary function can be written as a binary composition with another binary function and, in this way, all n-ary functions can be thought of as a kind of composition. Further, if any argument or parameter is outside the domain of the function, the result of that function application is simply not represented. Now, let us consider that all arguments and parameters are in the corresponding domain of the functions during their applications:

- The applications of unary functions, say there exist n_u such arithmetical functions, occupy n_u positions in any $\mathcal{F}(i,j)$ for all possible $i,j : \mathbb{N}$ and for each argument-parameter match. The corresponding real numbers that are represented are obtainable by applying these functions to all numbers in the previously defined cell (p) only, including the rational number that has been obtained by applying the above q equation, while, for some indexes i,j, some elements of $\mathcal{F}(i,j)$ are rational (e.g. $cos\ \pi = -1$). In the case where the first cell is being defined, this list is empty. Otherwise, for the unary functions only, $|\mathcal{F}(i,j)| = n_u \times |\mathcal{F}(a,b)|$ where $\langle a,b\rangle = h(g(i,j)-1)$ is a pair, as defined above.

- The application of binary functions, say there exist n_b such arithmetical functions, occupy the subsequent positions in the same list $\mathcal{F}(i,j)$, from the $(n_u \times |\mathcal{F}(a,b)|+1)$'th position on where $\langle a,b\rangle = h(g(i,j)-1)$ is a pair, as defined above. Since these functions are binary, for any cell c from the first cell on, we just pair p (previous cell) with all elements of the cell c from the first cell, $S(1,1)$, to the previous cell, p, if there exists such lists, and let the corresponding numeric representations be the last numbers of $\mathcal{F}(i,j)$ during the process. There may exist some rational numbers in this list, likewise the unary case but here as natural results from two real numbers. Further, we can represent other numbers by permuting these pairs, which are arguments, and, finally, for any cell, any value, we carry on counting the real numbers including both symmetric rational values. Finally, $\mathcal{F}(1,1)$ is empty and $\mathcal{F}(1,2)$ contains only representations of unary functions applications where the only argument is $S(1,1)_0$.

Again, the same procedure is applied with the symmetric rational numbers in $S(i,j)_0$ that have been written already, according to the indexes i,j.

Here, given that fractions with different numerators or denominators can be equivalent to a unique rational number (see above), a reasonable lemma is as follows: Given D, which is a denumerable set, and E such that $E \subset D$. If $D \setminus E$ is an infinite set, $D \setminus E$ is denumerable. Proof. Using a previous illustration, by removing the odd numbers from $\mathbb{N}$ we obtained a particular denumerable set, that is, the set of even numbers. On the other hand, by removing *one* element from any denumerable set, we obtain another denumerable set (by shifting all those subsequent numbers to one side). Thus, by removing any finite subset of a given countable set, we obtain a countable set. By repeating the removal process infinitely for an infinite subset, we shall obtain either a finite set (and this case is discarded here) or an infinite subset, that is, in accordance with the above illustration which produced even numbers, which in turn is regarded as denumerable. In the latter case, the short number of rules, for being general, uniformly yield that the infinite resulting sets are either denumerable or non-denumerable. However, since the illustration with the set of all odd numbers gives a denumerable set, $D \setminus E$ is denumerable only.

Note that the final ordering of the real numbers is not an ascending order in terms of magnitude, that is, there are some cases where $elem\langle\mathcal{F}(i,j),k\rangle \geq elem\langle\mathcal{F}(m,n),p\rangle$ for $elem\langle\mathcal{F}(i,j),k\rangle$ prior to $elem\langle\mathcal{F}(m,n),p\rangle$ in the final triangular ordering from S.

The obtained object does not really represent $\mathbb{R}$ since there are repetitions of elements in the representation. However, by applying this lemma removing the occurrences of all repetitions and fixed points, as we should do with fractions e.g. $1/2, 2/4, 3/6, ...$ for representing $\mathbb{Q}$, we obtain a proper and denumerable representation of all elements of $\mathbb{R}$. Finally, the present author has proved that $|\mathbb{R}| = |\mathbb{N}|$, which means that $\mathbb{R}$ is denumerable.

Finally, since a complex number of the form $a+b\sqrt{-1}$ is equivalent to a pair, $\langle a,b\rangle$ for $a,b \in \mathbb{R}$ for all such numbers, it is shown that $\mathbb{C}$ (and any such a set) is denumerable, e.g. by using the above triangular technique after the denumeration of $\mathbb{R}$.

As a concluding remark, Leibniz was absolutely right for having declared to be no good insisting on investigating the infinite, like Cantor did later, in 1891.

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